On the Vibration of Imperfect Circular Disks

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An analytical method is presented for vibration analysis of a rotating circular disk with distributed imperfections such as the nonuniform thickness, density, Young's modulus, Poisson's ratio and the odd distribution of internal stresses. In this paper, the linear governing equation of the disk is formulated, and then the relation between each imperfection and the split of degenerate modes is derived. In the derivation, the distributed imperfection, which is a periodic function of angular position, is expanded as a Fourier series. The derived simple relation suggests that the zeroth order imperfection causes the shift in the natural frequencies of all modes, whereas the 2m-th order harmonic imperfection results in the split of the natural frequencies, with the equal magnitudes but with different signs, for a pair of original degenerate modes with m nodal diameters.

Key Words: Imperfection, Split Modes, Degenerate System, Repeated Natural Frequency, Rotating Disk

Nomenclature

a	: Inner-clamping radius of the disk
A	: Disk area
b	: Outer radius of the disk
D	: Flexural rigidity of the disk
E	: Young's modulus of the disk
h	: Thickness of the disk
m, n	: Disk mode with m nodal diameters and n nodal circles
N_r, N_θ, $N_{r\theta}$	: Centrifugal stress resultants
$q_{mn}(t)$	: Generalized coordinates
r, θ, z	: Polar coordinates for the disk
T	: Kinetic energy
u	: Displacement in r direction
v	: Displacement in θ direction
V	: Potential energy
w	: Disk transverse displacement
α	: Imperfections of disk parameters
ρ	: Mass per unit volume of the disk
ν	: Poisson's ratio of the disk
ε	: Dimensionless frequency shifting caused by imperfections
σ_r, σ_θ, $\tau_{r\theta}$	: The normal and shear centrifugal stresses
ω	: Natural frequency (rad/s)
Ω	: Constant rotating speed of the disk (rad/s)

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1. Introduction

Rotating disks are important machine components widely used in many industrial applications: circular saw blades, turbine rotors, brake systems, fans, flywheels, gears, grinding wheels, precision gyroscopes, computer storage devices, etc. The dynamics of rotating disks have attracted a lot of research interest since the famous early researches (Kirchhoff, 1850; Lamb and Southwell, 1921; Southwell, 1922). In this paper, our primary concern is the influence of disk imperfections on the transverse vibration of rotating disks. Unavoidably, disk imperfections more or less exist in all real disks. In some situations, they are introduced intentionally (Yu and Mote, 1987). For convenience in discussion, the disk imperfections may be divided into three groups by the way they influence the energy elements (Tobias, 1957): strain imperfections, kinetic imperfections, and both strain and kinetic imperfections. The strain imperfections affect the strain energy but leave the kinetic energy undisturbed, such as Young's modulus, Poisson's ratio, and initial stress distribution. The kinetic imperfections affect the kinetic energy but leave the strain energy undisturbed, such as a variation of density. Dimensional irregularities normally produce both strain and kinetic imperfections.

The disk imperfections can have some interesting and important implications for the dynamic response of disks, such as introduction of additional natural frequencies due to mode splitting, and formation of vibrational modes fixed to the disk due to preferential orientation of the split modes. To avoid or make use of these phenomena, it is necessary to understand their forming mechanism. The literature addressing the analysis of split modes includes Zenneck (1899), Tobias and Arnold (1957), Tobias (1957), Williams and Tobias (1963), Williams (1966), Ewins (1969) and Efstathiades (1971). Recently, Stange and MacBain (1983) investigated experimentally the dual mode phenomena of a mistuned bladed disk by holographic interferometry. Honda, Matsuhisa and Sato (1985) found that the imperfection produced a much greater effect on the response near the resonance, but little effect on it off the resonance. Yu and Mote (1987) analyzed the effects of the asymmetry through perturbation of a variational formulation of the plate vibration problem. Rim and Lee (1993) obtained the modal parameters of an outer-clamped annular disk under arbitrary in-plane (self-equilibrating) force by using perturbation and Galerkin methods, and Rim, Kim and Lee (1992) further developed an identification method to estimate the arbitrary in-plane force along the clamped outer edge numerically and experimentally. Tseng and Wickert (1994a) discussed the vibration of an annular disk under the asymmetry of the bolted connections that are used to generate the "clamped" interior boundary by both the experimental and the theoretical means. Lee and Hong (1995) predicted the effect of a concentrated mass along a radial line on the free bending vibration of a circular plate by Rayleigh-Ritz method. Nayfeh et al. (1976) and Parker and Mote (1996) used perturbation techniques to investigate the influence of the deviation of the boundary of a disk from annular or circular domain to the natural frequencies and mode shapes. Tseng and Wickert (1994b) investigated the effects of eccentrical clamping on the natural frequencies and modes of a classical thin plate experimentally and through global discretization of the Kample quotient.

To the authors' knowledge, no work has addressed yet the relation between the split modes and the distributed imperfections such as the nonuniform thickness and density in a circular disk. In the present investigation, we derive the equations of motion for a rotating disk with typical distributed imperfections, then formulate the analytical relation between the distributed imperfections and the resulting split modes. The analytical method is developed under the assumption that the characteristics of the disk are not influenced by the imperfections except the natural frequencies and the locations of nodal diameters to investigate the mode splitting.

2. Review of Previous Works

2.1 Solution of perfect disks

The transverse displacement of a perfect flexible disk can be expressed, according to the expansion theorem, as

$$\begin{aligned} w(r,\theta,t) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[a_{mn}w_{mn}^{c}(r,\theta) + b_{mn}w_{mn}^{s}(r,\theta)]q_{mn}(t) \\ &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}R_{mn}(\beta_{mn}r)\cos m(\theta+\theta_{mn0})q_{mn}(t) \quad (1) \end{aligned}$$

where

$$w_{mn}^{c}(r,\theta)=R_{mn}(\beta_{mn}r)\cos m\theta$$

$$w_{mn}^{s}(r,\ \theta)=R_{mn}(\beta_{mn}r)\sin m\theta$$
$$(\beta_{mn}^{4}=\omega_{mn}^{2}\rho h/D)$$

are the orthonormal modes (with m nodal diameters and n nodal circles) of the disk, $q_{mn}(t)$ are the corresponding time-dependent generalized coordinates, a_{mn} and b_{mn} are the constants determining θ_{mn0}, and ρ, h and $D\left(=\frac{Eh^3}{12(1-\nu^2)}\right)$ are the mass per unit volume, thickness and flexural rigidity of the perfect disk, respectively.

As implied by Eq. (1), it has been well known that for each natural frequency ω_{mn}, there exist two corresponding orthogonal natural modes: the "cosine" and "sine" modes, except for $m=0$. In other words, those natural frequencies are repeated in the sense that their values become identical. The two orthogonal modes corresponding to a repeated natural frequency may be superposed, leading to a resultant mode with the orientation angle θ_{mn0} relative to the disk. For a perfectly symmetric disk, whether uniform or not, theory indicates, and experiment verifies, that there will be no fixed preferential orientation (θ_{mn0}) of the mode with respect to the disk (Rayleigh, 1945, pp. 363-366), unless the initial conditions in case of free vibration or the exciting force in case of forced vibration is fully specified.

2.2 Imperfect disks

If the axisymmetry of the disk is disturbed by any imperfections, some repeated natural frequencies may split into pairs of two distinct natural frequencies ω_p and ω_q ($\omega_p < \omega_q$), which are frequently referred to as the split natural frequencies or modes. The two split modes have preferential orientations with respect to the disk irrespective of the presence of the initial conditions and excitations. Zenneck (1899) has shown that the angular positions of the split modes coincide with those for which the natural frequency is either a minimum ω_p or a maximum ω_q (or derived from Rayleigh's principle). Thus, by attaching a small concentrated mass to a perfect disk, the mode ω_p will align itself with the mass eccentricity at its antinode, while the other mode ω_q will align the eccentricity at its node.

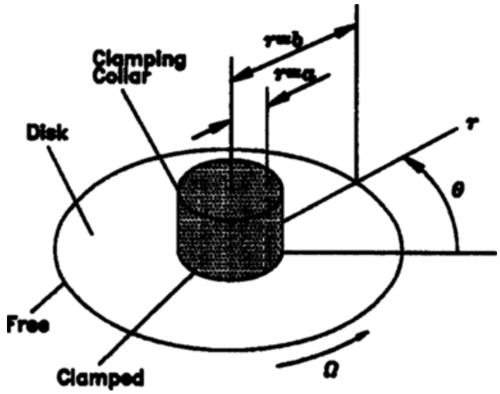

Fig. 1 Analysis model.

3. Analysis Model and Method

The analysis model of the rotating disk is shown in Fig. 1. The disk boundaries shown are inner-clamped and outer-free, but the analysis here is also valid for other boundary conditions. The classical thin plate theory or Kirchhoff plate theory is used to describe the disk vibration. The major features and limitations of the disk model are: the thick plate effects of rotary inertia and shearing deformations are not included; the linear theory is suitable; the normals to the middle plane of the plate are assumed to remain normal to the deflected middle plane during vibration; the dissipation due to damping is not included; the in -plane stresses of the disk due to centrifugal effects are included, but those due to thermal and residual stress effects are not included.

In the following sections, the governing equations for a disk with ideal, but typical distributed imperfections are formulated by Hamilton's principle and nondimensionalized. The distributed imperfections, which are periodic functions of angular position, are expanded as a Fourier series. And then the analytical relation between the imperfections and the split natural frequencies is derived and discussed. Finally, an example case is treated to illustrate the analytical development.

4. Energy Functions of Disk Vibration

Now, we consider the imperfections associated with the Young's modulus (E), the Poisson's ratio (ν), the plate's mass per unit volume (ρ), and the thickness of the disk (h). In addition, we also consider the imperfections associated with the stress resultants N_r, N_θ, and $N_{r\theta}$ given by

$$N_r=\int_{-h/2}^{h/2}\sigma_r\, dz,\quad N_\theta=\int_{-h/2}^{h/2}\sigma_\theta dz,$$
$$N_{r\theta}=\int_{-h/2}^{h/2}\tau_{r\theta} dz$$

where σ_r, σ_θ, $\tau_{r\theta}$ are the normal and shear centrifugal stresses in the polar coordinates, respectively.

The total kinetic energy due to transverse motion of the disk can be expressed as

$$T(t)=\frac{1}{2}\int_0^2\pi\int_a^b \rho h\dot{w}^2 rdrd\theta \quad (2)$$

where $w(r,\ \theta,\ t)$ is the disk transverse displacements. The kinetic energy due to rotation about its axis will be a constant for a constant rotating speed Ω, and it not included in this formulation since it will subsequently be canceled out when applying Hamilton's principle.

The potential energy of the disk includes three parts (Timoshenko and Gere, 1963). First, the energy V_0 due to deformation of the middle plane of the disk by forces applied in this plane (centrifugal stresses) is

$$V_0=\frac{1}{2}\iint_A\frac{1}{hE}[N_r^2+N_\theta^2-2\nu N_r N_\theta$$
$$+2(1+\nu)N_{r\theta}^2]\, rdrd\theta$$

where, A is the space area occupied by the disk. The stresses N_r, N_θ, and $N_{r\theta}$ are assumed to remain unchanged for small deflections, so the above strain energy remains constant during vibration, and thus we do not need to consider it for using Hamilton's principle. Second, the potential energy due to pure bending is

$$V_1(t)=\frac{1}{2}\int_0^{2\pi}\int_a^b\Big\{D(\Delta^2 w)^2-2D(1-\nu)$$
$$\frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)$$
$$+2D(1-\nu)\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\right]^2\Big\}rdrd\theta \quad (3)$$

where $\Delta^2 w=\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}$. Third, the potential energy due to the additional stretching of the middle plane in the presence of in-plane centrifugal forces is

$$V_2(t)=\iint_A\Big[N_r\frac{\partial u}{\partial r}+N_\theta\frac{\partial v}{r\partial\theta}$$
$$+N_{r\theta}\left(\frac{\partial u}{r\partial\theta}+\frac{\partial v}{\partial r}\right)\Big]rdrd\theta$$
$$+\frac{1}{2}\int_0^{2\pi}\int_a^b\Big[N_r\left(\frac{\partial w}{\partial r}\right)^2+\frac{N_\theta}{r^2}\left(\frac{\partial w}{\partial\theta}\right)^2$$
$$+2\frac{N_{r\theta}}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial\theta}\Big]rdrd\theta$$

where u and v represent the displacements in the r and θ directions of any points in the middle plane of the disk during vibration, respectively. For disk transverse vibration

$$u=v=0$$

Then

$$V_2(t)=\frac{1}{2}\int_0^{2\pi}\int_a^b\Big[N_r\left(\frac{\partial w}{\partial r}\right)^2+\frac{N_\theta}{r^2}$$
$$\left(\frac{\partial w}{\partial\theta}\right)^2+2\frac{N_{r\theta}}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial\theta}\Big]rdrd\theta \quad (4)$$

Note that $N_{r\theta}=0$ when the stress distribution is axisymmetric.

5. Equations of Motion

By applying Hamilton's principle, utilizing Eqs. (2) ~ (4), we obtain

$$\rho h\frac{\partial^2 w}{\partial t^2}+\Delta^2(D\Delta^2 w)-\frac{1}{r}\frac{\partial}{\partial r}\left(rN_r\frac{\partial w}{\partial r}\right.$$
$$-\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(N_\theta\frac{\partial w}{\partial\theta}\right)-\frac{2}{r}\frac{\partial}{\partial r}\left(N_{r\theta}\frac{\partial w}{\partial\theta}\right)$$
$$-\frac{2}{r}\frac{\partial}{\partial\theta}\left(N_{r\theta}\frac{\partial w}{\partial r}\right)-\frac{1}{r}\left(\frac{\partial w}{\partial r}+\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2}\right)\frac{\partial^2}{\partial r^2}$$
$$[D(1-\nu)]-\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2}{\partial\theta^2}[D(1-\nu)]$$
$$+\frac{2}{r^2}\left(\frac{\partial^2 w}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\frac{\partial^2}{\partial r\partial\theta}[D(1-\nu)]$$
$$-\frac{1}{r}\frac{\partial^2 w}{\partial r^2}\frac{\partial}{\partial r}[D(1-\nu)]$$
$$-\frac{2}{r^3}\left(\frac{\partial^2 w}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\frac{\partial}{\partial\theta}[D(1-\nu)]=0 \quad (5)$$

with associated boundary conditions.

It will prove convenient to introduce the dimensionless variables:

$$w^*=w^*/h_0,\quad r^*=r/b_0,$$
$$t^*=t/t_0,\quad \Omega^*=\Omega t_0,\quad t_0=b_0^2\sqrt{\rho_0 h_0/D_0}$$

and assume

$$E=E_0(1+\alpha_E),\quad \nu=\nu_0(1+\alpha_\nu),\quad \rho=\rho_0(1+\alpha_\rho)$$
$$h=h_0(1+\alpha_h),\quad N_r=N_{r0}(1+\alpha_{Nr})$$
$$N_\theta=N_{\theta 0}(1+\alpha_{N\theta}),\quad N_{r\theta}=\frac{\rho_0 h_0 b_0^2}{t_0^2}\alpha_{Nr\theta}$$

so that,

$$D=D_0(1+\alpha_D),\quad \alpha_D=\alpha_E+3\alpha_h\frac{2\nu_0^2}{1-\nu_0^2}\alpha_\nu$$
$$D(1-\nu)=D_0(1-\nu_0)(1+\alpha_{D\nu})$$
$$\alpha_{D\nu}=\alpha_E+3\alpha_h-\frac{\nu_0}{1+\nu_0}\alpha_\nu$$

where h_0, a_0, b_0, ρ_0, $\cdots$ are the parameters associated with the perfect disk, and α_h, α_a, α_b, α_ρ, $\cdots$ are the small quantities representing the imperfections which are functions of (r, θ). From physical view, α_h, α_a, α_b, α_ρ, $\cdots$ are the dimensionless deviations of the disk parameters from the idealized perfect disk. Note that in the above and following formulas, we ignore the small quantities of order ≥ 2. Then Eq. (5) can be rewritten, omitting asterisks for notational convenience, as

$$\begin{aligned}
&(1+\alpha_\rho+\alpha_h)\frac{\partial^2 w}{\partial t^2}+\Delta^2[(1+\alpha_D)\Delta^2 w]\\
&-\frac{1}{r}\frac{\partial}{\partial r}\left[(1+\alpha_{Nr})rN_{r0}\frac{\partial w}{\partial r}\right]\\
&-\frac{1}{r^2}\frac{\partial}{\partial\theta}\left[(1+\alpha_{N\theta})N_{\theta 0}\frac{\partial w}{\partial\theta}\right]\\
&-\frac{2}{r}\frac{\partial}{\partial r}\left(\alpha_{Nr\theta}\frac{\partial w}{\partial\theta}\right)-\frac{2}{r}\frac{\partial}{\partial\theta}\left(\alpha_{Nr\theta}\frac{\partial w}{\partial r}\right)\\
&-(1-\nu_0)\frac{1}{r}\left(\frac{\partial w}{\partial r}+\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2}\right)\frac{\partial^2\alpha_{D\nu}}{\partial r^2}\\
&-(1-\nu_0)\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2\alpha_{D\nu}}{\partial\theta^2}\\
&+(1-\nu_0)\frac{2}{r^2}\left(\frac{\partial^2 w}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\frac{\partial^2\alpha_{D\nu}}{\partial r\partial\theta}\\
&-(1-\nu_0)\frac{1}{r}\frac{\partial^2 w}{\partial r^2}\frac{\partial\alpha_{D\nu}}{\partial r}\\
&-(1-\nu_0)\frac{2}{r^3}\left(\frac{\partial^2 w}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\frac{\partial\alpha_{D\nu}}{\partial\theta}=0
\end{aligned}\tag{6}$$

6. Mode Splitting Caused by Imperfections

In this work, we do not include the boundary imperfections in the analysis due to mathematical difficulties. We assume a solution of the form

$$w_{mn}^c q_{mn}^c(t)=R_{mn}(r)\cos m(\theta-\theta_1)\cos\omega_1(t-t_1)$$
$$w_{mn}^s q_{mn}^s(t)$$
$$=R_{mn}(r)\sin m(\theta-\theta_2)\cos\omega_2(t-t_2)\tag{7a, b}$$

and

$$\begin{aligned}\omega_1^2&=(1+\varepsilon_1)\omega_0^2\\ \omega_2^2&=(1+\varepsilon_2)\omega_0^2\end{aligned}\tag{8}$$

where ω_0 is the repeated natural frequency of the corresponding perfect disk, the small quantities ε_1 and ε_2 are the constant frequency shifting caused by imperfections, and θ_1 and θ_2 are the preferential orientation angles of the split modes. $w_{mn0}=R_{mn}(r)(a_{mn0}\cos m\theta+b_{mn0}\sin\omega_0 t)$ is the corresponding solution of the perfect disk. The orientation angles θ_1 and θ_2 will be determined according to the fact that they make the split natural frequencies ω_1 and ω_2 either a minimum or a maximum (see discussion in Sec. 2. 2).

Substituting Eq. (7a) into Eq. (6), multiplying by $w_{mn}^c=R_{mn}(r)\cos m(\theta-\theta_1)$, integrating over disk area, and deleting small quantities of order ≥ 2, we obtain

$$\begin{aligned}
&\int_0^{2\pi}\int_a^b\Big[-(\varepsilon_1+\alpha_\rho+\alpha_h)\omega_0^2 w_{mn}^c+\Delta^2(\alpha_D\Delta^2\omega_{mn}^c)\\
&-\frac{1}{r}\frac{\partial}{\partial r}\left(\alpha_{N}rN_{r0}\frac{\partial w_{mn}^c}{\partial r}\right)-\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\alpha_{N\theta}N_{\theta 0}\frac{\partial w_{mn}^c}{\partial\theta}\right)\\
&-\frac{2}{r}\frac{\partial}{\partial r}\left(\alpha_{Nr\theta}\frac{\partial w_{mn}^c}{\partial\theta}\right)-\frac{2}{r}\frac{\partial}{\partial\theta}\left(\alpha_{Nr\theta}\frac{\partial w_{mn}^c}{\partial r}\right)\\
&-(1-\nu_0)\frac{1}{r}\left(\frac{\partial w_{mn}^c}{\partial r}+\frac{1}{r}\frac{\partial^2 w_{mn}^c}{\partial\theta^2}\right)\frac{\partial^2\alpha_{D\nu}}{\partial r^2}\\
&-(1-\nu_0)\frac{1}{r^2}\frac{\partial^2 w_{mn}^c}{\partial r^2}\frac{\partial^2\alpha_{D\nu}}{\partial\theta^2}\\
&+(1-\nu_0)\frac{2}{r^2}\left(\frac{\partial^2 w_{mn}^c}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial w_{mn}^c}{\partial\theta}\right)\frac{\partial^2\alpha_{D\nu}}{r\partial\theta}\\
&-(1-\nu_0)\frac{1}{r}\frac{\partial^2 w_{mn}^c}{\partial r^2}\frac{\partial\alpha_{D\nu}}{\partial r}-(1-\nu_0)\frac{2}{r^3}\left(\frac{\partial^2 w_{mn}^c}{\partial r\partial\theta}\right.\\
&\left.-\frac{1}{r}\frac{\partial w_{mn}^c}{\partial\theta}\right)\frac{\partial\alpha_{D\nu}}{\partial\theta}\Big]w_{mn}^c r\,dr\,d\theta=0
\end{aligned}\tag{9}$$

where the imperfection function can be expanded in Fourier series as

$$\alpha(r,\theta)=\alpha_0(r)+\sum_{n=1}^{\infty}\alpha_n(r)(A_n\cos n\theta+B_n\sin n\theta) \tag{10}$$

6.1 Imperfection in density

By substituting the expanded form of density imperfection $\alpha_\rho(r, \theta)$ into Eq. (9), we obtain

$$-\int_a^b\left[\pi\varepsilon_1+\pi\alpha_{\rho0}(r)+\frac{\pi}{2}A_{\rho2m}\cos 2m\theta_1\alpha_{\rho2m}+\frac{\pi}{2}B_{\rho2m}\sin 2m\theta_1\alpha_{\rho2m}(r)\right]\omega_0{}^2R^2_{mn}(r)\,rdr=0$$

or

$$\varepsilon_1=\varepsilon_{\rho0}-\frac{\Delta\varepsilon_{\rho1}}{2} \tag{11}$$

where,

$$\varepsilon_{\rho0}=-\frac{\int_a^b\alpha_{\rho0}(r)R^2_{mn}(r)\,rdr}{\int_a^b R^2_{mn}(rrdr} \tag{12}$$

$$\Delta\varepsilon_{\rho1}=(A_{\rho2m}\cos 2m\theta_1+B_{\rho2m}\sin 2m\theta_1)\cdot\frac{\int_a^b\alpha_{\rho2m}(r)R^2_{mn}(r)\,rdr}{\int_a^b R^2_{mn}(r)\,rdr} \tag{13}$$

Similarly, we obtain

$$\varepsilon_2=\varepsilon_{\rho0}+\frac{\Delta\varepsilon_{\rho2}}{2} \tag{14}$$

where $\Delta\varepsilon_{\rho2}$ has the same form as $\Delta\varepsilon_{\rho1}$ except that θ_1 be replaced by θ_2. The preferential angles θ_1 and θ_2 are determined from

$$\frac{d\Delta\varepsilon_{\rho1}}{d\theta_1}=0,\ \text{and}\ \frac{d\Delta\varepsilon_{\rho2}}{d\theta_2}=0 \tag{15}$$

Since except θ_1 and θ_2, $\Delta\varepsilon_{\rho1}$, and $\Delta\varepsilon_{\rho2}$ are completely the same, we have

$$\theta_1=\theta_2=\theta^* \tag{16}$$

so that

$$\Delta\varepsilon_{\rho1}=\Delta\varepsilon_{\rho2}=\Delta\varepsilon_\rho \tag{17}$$

and

$$\varepsilon_2-\varepsilon_1=\Delta\varepsilon_\rho$$

6.2 Imperfections in stresses

By substituting the expanded form of stress imperfections $\alpha_{Nr}(r, \theta)$, $\alpha_{N\theta}(r, \theta)$, $\alpha_{Nr\theta}(r, \theta)$ into Eq. (9), we obtain

$$\int_a^b\Big\{-\pi\varepsilon_1\omega_0{}^2R_{mn}-\frac{\pi}{r}\frac{\partial}{\partial r}\Big[\Big(\alpha_{Nr0}+\frac{1}{2}(A_{Nr2m}\cos 2m\theta_1+B_{Nr2m}\sin 2m\theta_1)\,\alpha_{Nr2m}\Big)rN_{r0}R'_{mn}\Big]$$
$$-\frac{N_{\theta0}}{r^2}\pi m^2\Big[-\alpha_{N\theta0}+\frac{1}{2}(A_{N\theta2m}\cos 2m\theta_1+B_{N\theta2m}\sin 2m\theta_1)\,\alpha_{N\theta2m}\Big]R_{mn}-\frac{m\pi}{r}\cdot$$
$$(A_{Nr\theta2m}\sin 2m\theta_1-B_{Nr\theta2m}\cos 2m\theta_1)\frac{\partial\alpha_{Nr\theta2m}}{\partial r}R_{mn}\Big\}R_{mn}rdr=0$$

Similarly, we have

$$\varepsilon_1=\varepsilon_{N0}-\frac{\Delta\varepsilon_N}{2}$$
$$\varepsilon_2=\varepsilon_{N0}+\frac{\Delta\varepsilon_N}{2}$$
$$\theta_1=\theta_2=\theta^*$$

where

$$\varepsilon_{N0}=\frac{-\int_a^b\left[\frac{1}{r}\frac{\partial}{\partial r}(\alpha_{Nr0}rN_{r0}R'_{mn})-m^2\frac{N_{\theta0}}{r^2}\alpha_{N\theta0}R_{mn}\right]R_{mn}rdr}{\omega_0^2\int_a^b R^2_{mn}rdr}$$

$$\Delta\varepsilon_N=(A_{Nr2m}\cos 2m\theta^*+B_{Nr2m}\sin 2m\theta^*)\cdot\frac{\int_a^b\frac{\partial}{\partial r}[r\alpha_{Nr2m}N_{r0}R'_{mn}]R_{mn}dr}{\omega_0^2\int_a^b R^2_{mn}rdr}$$
$$+m^2(A_{N\theta2m}\cos 2m\theta^*+B_{N\theta2m}\sin 2m\theta^*)\cdot\frac{\int_a^b\alpha_{N\theta2m}\frac{N_{\theta0}}{r}R^2_{mn}dr}{\omega_0^2\int_a^b R^2_{mn}\ rdr}$$
$$+2m(A_{Nr\theta2m}\sin 2m\theta^*-B_{Nr\theta2m}\cos 2m\theta^*)\cdot\frac{\int_a^b\alpha'_{N\theta2m}R^2_{mn}dr}{\omega_0^2\int_a^b R^2_{mn}rdr}$$

6.3 Imperfection in Young's modulus E ($\alpha_D=\alpha_E$, $\alpha_{D\nu}=\alpha_E$)

By substituting the expanded form of Young's modulus imperfection $\alpha_E(r, \theta)$ into Eq. (9), we obtain

$$\int_a^b\Big\{-\pi\varepsilon_1\omega_0^2R_{mn}+\pi\Delta_1^2(\alpha_{E0}\Delta_1^2R_{mn})+\frac{\pi}{2}(A_{E2m}\cos 2m\theta_1+B_{E2m}\sin 2m\theta_1)\cdot$$
$$\Delta_2^2(\alpha_{E2m}\Delta_1^2R_{mn})+\frac{2\pi m^2}{r^2}(A_{E2m}\cos 2m\theta_1$$

$$
\begin{aligned}
&+B_{E2m}\sin 2m\theta_1)\,\alpha_{E2m}\Delta_1^2 R_{no}\\
&-(1-\nu_0)\,\pi\frac{(rR'_{mn}-m^2R_{mn})\,\alpha''_{E0}+rR''_{mn}\alpha'_{E0}}{r^2}\\
&-(1-\nu_0)\,(A_{E2m}\cos 2m\theta_1\\
&+B_{E2m}\sin 2m\theta_1)\frac{\pi}{2}\Big[\frac{1}{r^2}(rR'_{mn}-m^2R_{mn})\,\alpha''_{E2m}\\
&-4m^2\frac{R''_{mn}}{r^2}\alpha_{E2m}-4m^2\frac{1}{r^3}(rR'_{mn}-R_{mn})\,\alpha'_{E2m}\\
&+\frac{R''_{mn}}{r}\alpha'_{E2m}+4m^2\frac{1}{r^4}(rR'_{mn}-R_{mn})\cdot\\
&\alpha_{E2m}\Big]\Big\}R_{mn}rdr=0
\end{aligned}
$$

where

$$\Delta_1^2 w=\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{m^2}{r^2}$$

$$\Delta_2^2 w=\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{5m^2}{r^2}$$

We have

$$\varepsilon_1=\varepsilon_{E0}-\Delta\varepsilon\frac{E}{2}$$

$$\varepsilon_2=\varepsilon_{E0}+\frac{\Delta\varepsilon_E}{2}$$

$$\theta_1=\theta_2=\theta^*$$

where

$$\varepsilon_{E0}=\frac{K_1}{\omega_0^2\int_a^b R_{mn}^2 rdr}$$

$$\Delta\varepsilon_E=\frac{K}{\omega_0^2\int_a^b R_{mn}^2 rdr}$$

$$
\begin{aligned}
K_1=&\int_a^b\Big[\Delta_1^2(\alpha_{E0}\Delta_1^2 R_{mn})-(1-\nu_0)\cdot\\
&\frac{(rR'_{mn}-m^2R_{mn})\,\alpha''_{E0}+rR''_{mn}\alpha'_{E0}}{r^2}\Big]R_{mn}rdr
\end{aligned}
$$

$$
\begin{aligned}
K=&(A_{E2m}\cos 2m\theta^*+B_{E2m}\sin 2m\theta^*)\cdot\\
&\int_a^b\Big\{-(\Delta_2^2(\alpha_{E2m}\Delta_1^2 R_{mn})-\frac{4m^2}{r^2}\alpha_{E2m}\Delta_1^2 R_{mn}\\
&+(1-\nu_0)\Big[\frac{1}{r^2}(rR'_{mn}-m^2R_{mn})\,\alpha''_{E2m}\\
&-4m^2\frac{R''_{mn}}{r^2}\alpha_{E2m}+\frac{R''_{mn}}{r}\alpha'_{E2m}-4m^2\frac{1}{r^3}\cdot\\
&(rR'_{mn}-R_{mn})\,\alpha'_{E2m}\\
&+4m^2\frac{1}{r^4}(rR'_{mn}-R_{mn})\,\alpha_{E2m}\Big]\Big\}R_{mn}rdr
\end{aligned}
$$

For α_ν and α_h, it is not difficult to get the similar results.

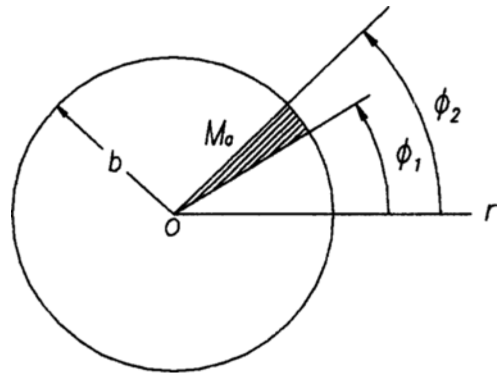

Fig. 2 A circular disk with mass attached.

7. An Example: A Circular Disk with a Mass Attached on a Radius

Consider vibration of a circular disk with a local imperfection in the form of a distributed mass along a radial line as shown in Fig. 2. Then, we have the expression

$$\rho=\rho_0(1+\alpha_\rho)$$

$$\alpha_\rho=\frac{2M_a}{\rho_0 b^2(\phi_2-\phi_1)}[H(\theta-\phi_1)-H(\theta-\phi_2)]$$

where Ma is the attached mass which is assumed to be small compared with that of the disk, and $H(\theta)$ is the Heaviside unit step function. The Fourier series expansion of α_ρ can be written as

$$\alpha_\rho(\theta)=\alpha_0+\sum_{n=1}^{\infty}(A_n\cos n\theta+B_n\sin n\theta)$$

where

$$
\begin{aligned}
\alpha_0=&\frac{1}{2\pi}\frac{2M_a}{\rho_0 b^2(\phi_2-\phi_1)}\int_0^{2\pi}[H(\theta-\phi_1)\\
&-H(\theta-\phi_2)]d\theta=\frac{M_a}{\pi\rho_0 b^2} \qquad (18)\\
A_n=&\frac{2}{2\pi}\frac{2M_a}{\rho_0 b^2(\phi_2-\phi_1)}\int_0^{2\pi}[H(\theta-\phi_1)\\
&-H(\theta-\phi_2)]\cdot\cos n\theta d\theta\\
=&\frac{2M_a}{\pi\rho_0 b^2(\phi_2-\phi_1)}\frac{(\sin n\phi_2-\sin n\phi_1)}{n}\\
B_n=&\frac{2}{2\pi}\frac{2M_a}{\rho_0 b^2(\phi_2-\phi_1)}\int_0^{2\pi}[H(\theta-\phi_1)\\
&-H(\theta-\phi_2)]\cdot\sin n\theta d\theta\\
=&\frac{2M_a}{\pi\rho_0 b^2(\phi_2-\phi_1)}\frac{\cos n\phi_1-\cos n\phi_2}{n}
\end{aligned}
$$

The origin of the polar coordinates can be chosen arbitrarily, hence without loss of generality, we consider the mass along $\theta=0$ line in polar coordinates. In the limiting case where ϕ_2 approaches ϕ_1, we have the asymptotic behavior, since $\phi_0\equiv(\phi_1+\phi_2)/2=0$, given by

$$A_n=\frac{2M_a}{\pi\rho_0 b^2}, \text{ and } B_n=0 \tag{19}$$

Inserting Eqs. (18) and (19), and $\alpha_n=1$ into Eqs. (11) and (13), and utilizing Eq. (16), we obtain

$$\varepsilon_{\rho 0}=-\frac{M_a}{\pi\rho_0 b^2}$$
$$\Delta\varepsilon_\rho=\frac{2M_a}{\pi\rho_0 b^2}\cos 2m\theta^*$$

From Eq. (15), we obtain

$$\theta^*=0$$

then

$$\varepsilon_1=-\frac{2M_a}{\pi\rho_0 b^2}, \text{ and } \varepsilon_2=0$$

Using Eq. (8), we have

$$\begin{aligned}\omega_1^2&=\left(1-\frac{2M_a}{\pi\rho_0 b^2}\right)\omega_0^2\\ \omega_2^2&=\omega_0^2\end{aligned} \tag{20}$$

The same problem was solved by Lee and Hong (1995) using Rayleigh-Ritz method and their results were

$$\omega_1^2=\left(1+\frac{2M_a}{\pi\rho_0 b^2}\right)^{-1}\omega_0^2\approx\left(1-\frac{2M_a}{\pi\rho_0 b^2}\right)\omega_0^2$$
$$\omega_2^2=\omega_0^2$$

which are identical to Eq. (20). Note that Eq. (20) holds for all modes with nonzero nodal diameters.

Figure 3 shows the typical mode shapes with

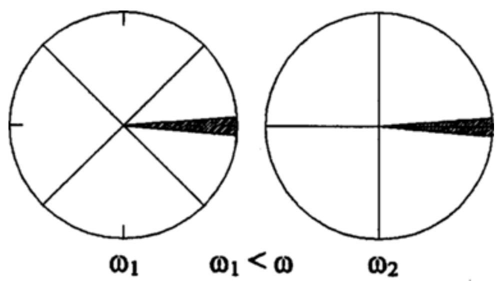

Fig. 3 Split mode configurations for (2, 0) mode.

split modal frequencies. The lines in the figure show the nodal diameters of the split modes, and the hatched area represent the attached mass.

From the theoretical results in Sec. 6 and the example above, one sees that the zeroth order imperfection $\alpha_0(r)$ will cause the shift in natural frequencies of all modes, but it does not cause the split of degenerate modes. The 2m-th harmonic component $A_{\rho 2m}\cos 2m\theta+B_{\rho 2m}\sin 2m\theta$ of the distributed imperfections will cause the split, with the equal magnitudes but with different signs, of the natural modes with m nodal diameters. The preferential angles for a pair of split modes are of the same values.

8. Discussion and Conclusions

In this work, we derived the analytical relation between the distributed imperfections and the split modes and a simple rule is obtained for predicting the split modes. A degenerate mode with $m(\neq 0)$ nodal diameters in the perfect disk split into a pair of modes with slightly different natural frequencies if there exists the 2m-th order harmonic component in the disk parameters. Similar behaviors have been identified for bladed disks (Ewins, 1969), circular saw blades with radial slots (Yu and Mote, 1987), bolted plates (Tseng and Wickert, 1994a), nearly annular or circular plates (Nayfeh, 1976; Parker and Mote, 1996), and eccentrically clamped annular plate (Tseng and Wickert, 1994b).

The values of the natural frequencies and the eigenfunctions of the imperfect disk are determined from the distribution of the imperfections. The zeroth order imperfection ($\alpha_0(r)$ in the Fourier series) causes the shift in the natural frequencies of all modes, but not splitting. Whereas the 2m-th order harmonic imperfection results in the split of the natural frequencies, with the equal magnitudes but with different signs, for a pair of original degenerate modes with m nodal diameters. The preferential angles for a pair of split modes are of the same values.

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